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Existence and Non-existence of a Fast Diffusion Equation in R^n

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In this paper we study the global existence and asymptotic behaviour of solutions to $u_t = \Delta \log u$ for the Cauchy initial value problem in R^n. We prove that if [illegible]

[illegible] $p > n/2$. Hence, we extend a previous result of Vazquez [19] which claims that $\int_{R^n} u\, dx = \infty$ for $0 < t < t_{\max}$.

1. INTRODUCTION

The differential equation

$$u_t = \Delta \log u \tag{1}$$

arises from many physical, as well as geometrical problems. In particular, it models cross-field convection diffusion of plasma, the expansion of a thermalized electron cloud in physics [4] and Ricci flow in two dimensional Riemannian geometry [20]. It also relates to the central limit approximation of the Boltzmann equation.

In this paper we shall study the global existence, finite time extinction and asymptotic behaviour of solutions to (1) for Cauchy problem in R^n,

$$\text{(I)} \quad \begin{cases} u_t = \Delta \log u, & x \in R^n, \quad t > 0, \\ u(x, 0) = u_0(x) > 0, & x \in R^n. \end{cases} \tag{2}$$

We assume that $u_0(x)$ is a continuous function.

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The equation (1) is the limiting case of $u_t = \Delta u^m$ as $m \to 0+$, which was demonstrated in [16], [20]. Therefore it is interesting to compare some known results on (I) to that of well studied Cauchy problem of fast-diffusion equation

$$\text{(MI)} \quad \begin{cases} u_t = \Delta u^m, & x \in R^n, \quad t > 0, \\ u(x, 0) = u_0(x) > 0, & x \in R^n, \end{cases} \tag{3}$$

where $0 < m < 1$. It is known that (MI) is a well-posed problem for $(n-2)_+/n < m < 1$. To be more specific, consider the case $u_0(x) \in L^1(R^n) \cap L^\infty(R^n)$, then there exists a unique function $u \in C([0, \infty); L^1(R^n)) \cap L^\infty(R^n \times (0, \infty))$ which satisfies (MI) in a weak sense and such that $u(0) = u_0$, see [2], [5]. Moreover, the solution u depends continuously on u_0 in the L^1-norm and the maps $S_t \mapsto u(\cdot, t)$ form a semigroup of order-preserving contraction in $L^1(R^n)$. The fact that $u(\cdot, t) \in L^1(R^n)$ is called finite mass property of u due to the reason that u is a density function in the corresponding physical models.

We give a brief account of known results on (I) before we proceed to elaborate on the different behaviour of solutions of (I) from that of (MI). The study of (I) and related equations was conducted extensively by Esteban, Rodriquez and Vazquez [13], Herrero [14], Rodriguez and Vazquez [17] and [19], where many important results were established such as necessary and sufficient conditions for existence, regularity of solutions and the relation of equations to porous media equations. It was also studied by Hui [16] to show the non-existence of fundamental solutions and by Wu [20] in connections with two dimensional Ricci flow.

Surprisingly, whether (I) has finite mass solution depends on the dimension of space as revealed by the study of Vazquez and his collaborators. Indeed, it was proved in [13] by Esteban, Rodriquez and Vazquez that there exist such solutions when $n = 1$. Nonetheless, it was proved in [19] (see also [16]) that there does not exist a solution of (I) which satisfies $u(\cdot, t) \in L^1(R^n)$ for any $t \in (0, t_{\max})$, where $t_{\max}$ is the maximal existence time, when $n \geqslant 3$. In fact, it was proved in [19] that for every solution u there holds

$$\int_{R^n} u(x, t)\, dx = \infty, \qquad 0 < t < t_{\max}.$$

It is a strong contrast to the case $n = 1$, where (I) has global solutions with finite mass. For $n = 2$ there is a solution which has finite mass and becomes extinct in finite time. It takes the form [19],

$$u(x, t) = \frac{8(T - t)}{(1 + |x|^2)^2}.$$

Whether there exists a global solution with finite mass when $n=2$ is still open. But, our result (see Theorem 1 below) suggests that the finite mass global solution does not exist when $n=2$. The reason why for $n\geqslant 3$ (I) has no finite mass solution is that if $u_0(x)\in L^1(R^n)$ then the decay as $|x|\to\infty$ is too fast to stop solution from going to zero instantaneously in the whole space R^n. This will be clear from the proof of our Theorem 1. A weaker but quicker result is that if u_0 has sufficient decay as $|x|\to\infty$, then the solution u tends to zero (or becomes extinct) in finite time. This can be shown by using maximum principle and the explicit solution,

$$u(x,t)=\begin{cases}\dfrac{8(T-t)}{(1+|x|^2)^2}, & n=2;\\[2ex] \dfrac{(T-t)^{n/(n-2)}\,2(n-2)\lambda}{2(n-2)+\lambda(T-t)^{2/(n-2)}\,|x|^2}, & \lambda>0,\quad n\geqslant 3.\end{cases} \tag{4}$$

Given the situation that for $n\geqslant 3$ there does not exist a solution of (I) which belongs to $L^1(R^n)$, the natural question to ask is: what is the appropriate L^p space that a solution belongs? In particular, will the integrability property be different for finite time extinction solution and global solution? It turns out that these are subtle questions to answer. We prove that there exist global solutions which have the same integrability as the explicit solution in (4). On the other hand, there exists no solution, local or global, with stronger integrability. We summarize our main result in the following theorem.

THEOREM 1. *Let $n\geqslant 3$.*

(a) *There exists a non-trivial global solution u of (I) which has the property that $u(\cdot,t)\in L^p(R^n)$, for any $p>n/2$ and for all $t>0$. But it does not belong to any $L^p(R^n)$ if $p\leqslant n/2$.*

(b) *For any $T>0$, there exists a solution $u(\cdot,t)$ which has a finite maximal existence time T and such that for any $p>n/2$, $u(\cdot,t)\in L^p(R^n)$, $0<t<T$. Moreover, u becomes extinct at $t=T$, i.e., $u(\cdot,T)\equiv 0$.*

(c) *Any non-trivial solution u satisfies*

$$\int_{R^n} u^p(x,t)\,dx=\infty \qquad 0<t<t_{\max}, \tag{5}$$

where $1<p\leqslant n/2$, and $t_{\max}>0$ is the maximal existence time.

Remark. Our result not only extends the previous results of [16], [19], but it also gives the optimal integrability condition for local and global existence of solutions.

The structure of this article is as follows. In Section 2 we shall construct a family of global solutions which have certain structure invariant properties, and at the same time satisfy the integrability conditions in (a) of Theorem 1. The explicit solution we demonstrated in (4) fulfils the claim in (b). It is a special case of a family of self-similar solutions which exist locally in time and become extinct in finite time $T>0$. We shall study their existence and the asymptotic properties in Section 3. In Section 4 we prove Theorem 1.

2. THE EXISTENCE OF GLOBAL SOLUTIONS

In this section we prove (a) of Theorem 1. Due to the scaling invariant property of (1) we shall look for global self-similar solutions. It is our belief that these special solutions form the global attractor of the heat equation (1), though we cannot prove it now. For simplicity we seek self-similar solution of equation (1) which takes the form

$$u(x, t)=e^{-t}v(|x|/e^{-t/2}).$$

It is easy to see that v satisfies the equation

$$v\left(v''+\frac{n-1}{r}\,v'\right)-(v')^2+\left(v+\frac{r}{2}\,v'\right)v^2=0 \tag{6}$$

$$v'(0)=0, \qquad v(0)>0,$$

where $r=|x|/e^{-t/2}$, $v'=dv/dr$. We note that the integrability of global self-similar solutions is directly related to the asymptotic behaviour of v as $r\to\infty$.

Lemma 1. *Let* $n\geqslant 3$. *Then every solution of* (6) *is positive and monotone decreasing on* $[0,\infty)$. *Moreover, the following inequalities hold,*

$$v'+\frac{r}{2}\,v^2>0, \qquad v+\frac{r}{2}\,v'>0, \qquad \textit{for} \quad r>0.$$

Proof. Divide (6) by v^2 and integrate on $[0, r]$, we have

$$r^{n-1}\left(v'+\frac{r}{2}\,v^2\right)=v\int_0^r s^{n-1}\left(\frac{n}{2}-1\right)v\,ds. \tag{7}$$

Suppose v is not positive on $[0, \infty)$. Let $r = r_0 > 0$ be the first point where $v = 0$. Then, it is clear from (7) that

$$\lim_{r \to r_0} r^{n-1} v'/v = \int_0^{r_0} s^{n-1} \left(\frac{n}{2} - 1 \right) v \, ds > 0. \tag{8}$$

In particular, v'/v is positive for all r close to r_0. But, this is impossible. Hence v is positive, and consequently $v' + (r/2) v^2 > 0$ on $[0, \infty)$. Then it is easy to verify that $v' < 0$ on $[0, \infty)$. Let

$$g = v + \frac{r}{2} v'.$$

Then, $g(0) = v(0) > 0$,

$$\begin{aligned} g' &= \frac{3}{2} v' + \frac{r}{2} v'' \\ &= -\frac{r}{2} g v + \frac{4-n}{2} v' + \frac{r}{2} \frac{(v')^2}{2} \\ &= \left(-\frac{r}{2} v + \frac{4-n}{r} \right) g + \frac{1}{2rv} [2(n-4) v^2 + r^2 (v')^2]. \end{aligned} \tag{9}$$

If $g = 0$ at $r > 0$, then $v' = -2v/r$ and

$$\begin{aligned} g' &= \frac{1}{2rv} [2(n-4) v^2 + r^2 (v')^2] \\ &= \frac{(n-2)}{r} v > 0. \end{aligned} \tag{10}$$

A contradiction! Therefore, $g > 0$ on $[0, \infty)$. This completes the proof of Lemma 1.

Remark. It is clear from the proof of Lemma 1 that for the case $n = 2$ we can write out the solutions of (6) explicitly. They take the form

$$v(r) = \frac{4v(0)}{4 + r^2 v(0)}.$$

The corresponding solution of (1) is:

$$u(x, t) = \frac{4C}{4e^t + C |x|^2},$$

where $C > 0$ is a constant. It is worth to point out that the above explicit global solution for $n = 2$ has the property that it is in $L^p(R^2)$ for any $p > 1$, but it is not in $L^1(R^2)$.

LEMMA 2. *Let v be a solution of* (6) *and* $n \geqslant 3$. *Then*

$$\frac{v}{r^2 \log r} \to 4(n-2) \qquad \textit{as} \quad r \to \infty.$$

Proof. Let $w = r^2 v$, then

$$v' = w' r^{-2} - 2wr^{-3}, \qquad v'' = [w'' - 4w' r^{-1} + 6wr^{-2}] r^{-2}.$$

Elementary calculation yields

$$w'' = \frac{2(n-1) + w}{2r} w' - \frac{(w')^2}{w} + \frac{2(2-n)}{r^2} w = 0. \tag{11}$$

We claim that $w \to \infty$ as $r \to \infty$. For otherwise, since $w' = r(rv' + 2v) > 0$ by Lemma 1, $w \to C > 0$ as $r \to \infty$. Then there exist $M > 1$ and $K > 0$ such that

$$w'' + \frac{M}{r} w' > \frac{K}{r^2} \qquad \text{for} \quad r \gg 1.$$

An integration gives

$$w' r^M > \frac{K}{2(M-1)} r^{M-1} \qquad \text{for} \quad r \gg 1.$$

In turn $w \to \infty$ as $r \to \infty$. A contradiction! This proves the claim. Consequently $w > C \log r$ for $r \gg 1$ and $C > 0$.

On the other hand, $v' < 0$ implies $(w')^2 < 4w^2/r^2$. Consequently we get from (11)

$$w'' + \frac{2(n-1) + w}{2r} w' < \frac{2n}{r^2} w.$$

An integration of the above inequality on $[0, r]$, after it being multiplied by $\rho(r) = \exp(\int_0^r w(s)/2s\, ds)$, gives

$$w' r^{n-1} \rho(r) < 2n \int_0^r s^{n-3} \rho(s)\, w(s)\, ds.$$

It is easy to verify using L'Hospital's rule that

$$\frac{\int_0^r s^{n-3}\rho(s)\, w(s)\, ds}{r^{n-2}\rho(r)} = 2.$$

Hence, for $r \gg 1$ and $\varepsilon > 0$,

$$w' r < (4n + \varepsilon) \qquad \text{and} \qquad w < (4n + \varepsilon) \log r. \tag{12}$$

In short, w behaves like $\log r$ for $r \gg 1$. We give more precise description in what follows. An integration of (11), after it being multiplied by $\rho(r) = \exp(\int_0^r w(s)/2s\, ds)$, on $[0, r]$ gives

$$w' r^{n-1}\rho(r) = 2(n-2)\int_0^r s^{n-3}\rho(s)\, w(s)\, ds + \int_0^r s^{n-1}\rho(s)\, \frac{(w')^2}{w(s)}\, ds.$$

Another applications of L'Hospital's rule combined with (12) then yields

$$w' r \to 4(n-2) \qquad \text{as} \quad r \to \infty.$$

A more detailed calculation implies

$$\int_0^r s^{n-3}\rho(s)\, w(s) = 2r^{n-2}\rho(r) - 2(n-2)\int_1^r s^{n-3}\rho(s)\, ds,$$

$$\int_0^r s^{n-3}\rho(s)\, ds = 2\,\frac{\rho(r)}{w(r)}\, r^{n-2} - 2(n-2)\int_1^r s^{n-3}\, \frac{\rho(s)}{w(s)}\, ds$$

and

$$\frac{\int_1^r s^{n-3}\, \dfrac{\rho(s)}{w^2(s)}\, ds}{r^{n-2}\rho(r)/w} \to 2.$$

Therefore

$$w(w' r - 4(n-2)) = -8(n-2)^2 + O\left(\frac{1}{\log r}\right).$$

Thus,

$$\frac{w}{\log r} \to 4(n-2) \qquad \text{as} \quad r \to \infty.$$

This completes the proof of lemma.

Remark. It is interesting to observe that for any $t>0$, the global self-similar solution u_G we construct in this section has the asymptotic behaviour

$$u_G(x,t) \sim \frac{\log|x|}{|x|^2} \qquad \text{as} \quad x \to \infty,$$

whereas the explicit finite time extinction solution u_L in (4) has the asymptotic behaviour

$$u_L(x,t) \sim \frac{1}{|x|^2} \qquad \text{as} \quad x \to \infty.$$

Their difference in asymptotic behaviour as $x \to \infty$ is very subtle and we cannot distinguish them using L^p space. In fact, they both belong to $L^p(R^n)$ for $p>n/2$ and they are both non-integrable in space $L^p(R^n)$ for $p \leqslant n/2$.

3. THE FINITE TIME EXTINCTION SOLUTION

Let $T>0$ be fixed. Suppose $u(x,t)$ is a solution of (1) which becomes extinct at $t=T$. Then we can perform a self-similar transformation

$$u(x,t) = (T-t)^{\alpha}\, v(y,s) \qquad y = \alpha^{1/2} x (T-t)^{\beta},$$
$$s = -\alpha \log(T-t) + \alpha \log T,$$

where $\alpha = 1+2\beta$, $\beta>0$ is arbitrary. The equation for v is:

$$v_s = \Delta \log v + v + ky \cdot \nabla v, \tag{13}$$

where $k=\beta/\alpha<1/2$. (13) will prove to be useful in Section 4.

For the present, we shall only consider the radial symmetric steady-state solutions of (13) which take the form $v(y)=v(r)$, where $r=|y|$, and satisfy

$$\begin{aligned} &v\left(v'' + \frac{n-1}{r} v'\right) - (v')^2 + (v+krv')v^2 = 0, \\ &v'(0)=0, \qquad v(0)=\eta>0. \end{aligned} \tag{14}$$

Here $0<k<1/2$.

The main result of this section is to prove that every solution of (14) is a ground state, i.e., it is positive on $[0,\infty)$, and it has a uniform asymptotic behaviour as $r \to \infty$.

THEOREM 2. *Let $n \geqslant 3$ and $0 < k < 1/2$. Then every solution of* (14) *is a ground state. Moreover,*

$$\lim_{r \to \infty} r^2 v(r) = \frac{2(n-2)}{1-2k}. \tag{15}$$

Remark. The above result shows that the asymptotic behaviour of finite time extinction self-similar solutions is different from that of global existence counterpart by a mere $\log |x|$, which cannot be distinguished using L^p setting.

We shall prove Theorem 2 by establishing several lemmas and propositions. As we shall see that the two cases: $0 < k < 1/n$ and $k > 1/n$ are quite different. We start by proving the following proposition.

PROPOSITION 1. *For each $\eta > 0$, $v(r; \eta)$ is a monotone decreasing ground state. Furthermore, there exists $c > 0$ such that $v(r) \geqslant cr^{-1/k}$ for all $r \gg 1$.*

Proof. Let $h = v + krv'$. Then it can be demonstrated as in the proof of Lemma 1 that $h > 0$ for $r > 0$. Consequently, v is a ground state. It is then easy to verify that $v' < 0$ for $r > 0$. An integration of $h > 0$ gives the last conclusion. This completes the proof of proposition.

LEMMA 3. *Let $n \geqslant 3$ and $1/n < k < 1/2$. Then every solution of* (14) *satisfies* (15).

Proof. Multiplying (14) by r^{n-1} and integrating on $[0, r]$, one finds

$$r^{n-1}(v' + krv^2) = (kn-1)v \int_0^r s^{n-1} v(s)\, ds \geqslant \frac{kn-1}{n} v^2(r) r^n. \tag{16}$$

An integration of the above then yields

$$v(r) > \frac{2nv(0)}{r^2 v(0) + 2n}. \tag{17}$$

We observe that $\bar{v} = 2(n-2)/(1-2k)r^2$ is a solution of (14) which is singular at $r = 0$. It is clear that for each $\eta > 0$, $v(r; \eta) < \bar{v}(r)$ if $r \ll 1$. We prove that this inequality holds for all $r > 0$. Suppose the contrary. Let r_1 be the first point where $v = \bar{v}$. For $r \in (0, r_1]$, there hold

$$\begin{aligned} r^{n-1}(v' + krv^2) &= (kn-1)v \int_0^r s^{n-1} v(s)\, ds, \\ r^{n-1}(\bar{v}' + kr\overline{v^2}) &= (kn-1)\bar{v} \int_0^r s^{n-1} \bar{v}(s)\, ds. \end{aligned} \tag{18}$$

It is immediate that $z = \bar{v} - v$ satisfies

$$z' + kr(v + \bar{v})z > 0 \qquad 0 < r \leqslant r_1,$$

since $k > 1/n$. One finds, after an integration of the above that $\bar{v}(r_1) > v(r_1)$. A contradiction. Thus, $v < \bar{v}$ for $r > 0$.

Let $G = v + rv'/2$, we prove $G > 0$ for $r > 0$. We observe that if $G(r) = 0$, then

$$G'(r) = \frac{n-2}{r} v(r) + \frac{2k-1}{2} rv^2(r) > 0$$

since $v < \bar{v}$. This, combined with $G > 0$ for $r \ll 1$, implies that $G > 0$ for $r > 0$. Hence, $r^2 v(r)$ is a bounded and increasing positive function.

Let $w = r^2 v$, then one finds after an elementary calculation that

$$w'' + \frac{(n-1) + kw}{r} w' - \frac{(w')^2}{w} + \frac{2(2-n) + (1-2k)w}{r^2} w = 0. \tag{19}$$

It is easy to deduce that $w \to 2(n-2)/(1-2k)$. This completes the proof of lemma.

Proposition 2. *Let* $0 < k < 1/n$. *Then* $r^2 v < 2n$.

Proof. One gets from (16) that

$$-r^{n-1}(v' + krv^2) = (1-kn)v \int_0^r s^{n-1} v(s)\, ds \geqslant \frac{1-kn}{n} v^2(r) r^n.$$

Therefore

$$-\frac{v'}{v^2} \geqslant \frac{r}{n}.$$

An integration then yields

$$v(r) \leqslant \frac{2nv(0)}{v(0)r^2 + 2n} < \frac{2n}{r^2}.$$

Lemma 4. *Let* $n \geqslant 3$ *and* $0 < k < 1/n$. *Then for every solution of* (14) *there exists* $c > 0$ *such that for all* $r \geqslant 1$, $r^2 v > c$.

Proof. Let $z(t)=w(r)$, $t=\log r$, where $w(r)=r^2v(r)$ satisfies (19). Then z is a solution of the following autonomous system:

$$z''+z'(n-2+kz)+[2(2-n)+(1-2k)z]z-\frac{(z')^2}{z}=0. \tag{20}$$

Let

$$E(t)=\frac{1}{2}\left(\frac{z'}{z}\right)^2+2(2-n)\log z+(1-2k)z.$$

Then, using the equation (20), we get

$$E'(t)=-\left(\frac{z'}{z}\right)^2(n-2+kz)\leqslant 0.$$

Consequently, there is no periodic solution of (20).

Apparently z is an increasing function for t near to $-\infty$. If z is increasing for all $t\in(-\infty,\infty)$ then we are done. Otherwise at the first zero $t=t_1$ of z', we have

$$\frac{2(n-2)}{1-2k}<z<2n.$$

Since $2(2-n)\log z+(1-2k)z$ is a convex function for $z>0$, we get

$$\begin{aligned}
&E(t_1)\leqslant C_1\\
&\quad C_1=\max(2(2-n)\log 2n+(1-2k)\,2n,\,2(2-n)\\
&\qquad\times\log\left(\frac{2(n-2)}{1-2k}\right)+2(n-2)\Bigg).
\end{aligned}$$

For $t>t_1$, $E(t)\leqslant E(t_1)\leqslant C_1$ gives

$$-2\log z+(1-2k)z\leqslant C_1.$$

Thus, $z\geqslant e^{-C_1/2}$.

Proof of Theorem 2. If $k=1/n$, then $2(n-2)/(1-2k)=2n$ and

$$v(r)=\frac{2nv(0)}{v(0)r^2+2n}.$$

This is the explicit solution given at the beginning of the paper.

If $k>1/n$ then it follows from Lemma 3.

For the case of $k<1/n$, Proposition 2 and Lemma 4 imply that $r^2 v$ is bounded from both above and below. In addition, the autonomous system (20) has only one equilibrium point and no periodic orbit in the region $c<z<2n$ for any $c>0$. Hence,

$$z(t) \to \frac{2(n-2)}{1-2k} \qquad \text{as} \quad t \to \infty.$$

This completes the proof of theorem.

4. THE NON-EXISTENCE

In this section we prove Theorem 1. In particular, Theorem 1(c) will be established by the following two lemmas.

LEMMA 5. *Let $n \geqslant 3$ and $u_0>0$ be a bounded, radially symmetric and non-increasing function. If $u_0 \in L^{n/2}(R^n)$, then $u(x,t)$ cannot be a global solution. Moreover, there exists $T>0$ such that $u(\cdot, T) \equiv 0$.*

Proof. Suppose to the contrary that u is a global solution. It is clear that u is uniformly bounded by $\|u_0\|_\infty$. In addition, it is radially symmetric and non-increasing. Hence, for any $t>0$, supp $u(\cdot, t)$ is either a ball or the whole space. But by the result of Vazquez [19], $\|u(t)\|_1 = \infty$ for $t>0$. Therefore the support of $u(\cdot, t)$ is the whole space for $t>0$. By the standard parabolic theory, u is a smooth function for $(x,t) \in R^n \times (0, \infty)$, see also [19].

Let $l>0$ and $\phi_l \in C_0^\infty(R^n)$ with the following properties:

(i) $\phi_l(x) \equiv 1$ for $|x| \leqslant l$, $\phi_l(x) \equiv 0$ for $|x| \geqslant 2l$;

(ii) $|\nabla \phi_l| \leqslant c/l$, $|\Delta \phi_l| \leqslant c/l^2$,

where c is a constant independent of l.

Define

$$v(t;l) = \int_{R^n} u^{n/2}(x,t)\, \phi_l(x)\, dx.$$

Then we find, by using equation (1)

$$v'(t;l) = \frac{n}{2} \int_{R^n} u^{n/2-2}(x,t)\, \phi_l(x)\, \Delta u\, dx - \int_{R^n} \phi_l(x)\, |\nabla u|^2\, u^{n/2-3}\, dx,$$

which, upon an integration by parts, yields, for $t>0$,

$$v'(t;l)=\frac{n}{n-2}\int_{R^n} u^{(n-2)/2}(x,t)\,\Delta\phi_l(x)\,dx$$

$$-\frac{n(n-2)}{4}\int_{R^n}\phi_l(x)\,|\nabla u|^2\,u^{n/2-3}\,dx. \tag{21}$$

By Holder's inequality

$$\begin{aligned}
&\int_{R^n} u^{(n-2)/2}(x,t)\,|\Delta\phi_l(x)|\,dx\\
&\leqslant\left(\int_{R^n\setminus B_l} u^{n/2}\phi_l\,dx\right)^{(n-2)/n}\left(\int_{R^n}|\Delta\phi_l|^{n/2}/\phi_l^{(n-2)/2}\,dx\right)^{2/n}\\
&\leqslant C(n)\left(\int_{R^n\setminus B_l} u^{n/2}\phi_l\,dx\right)^{(n-2)/n},
\end{aligned}$$

where B_l is a ball with radius l centered at origin. Hence,

$$v'(t;l)\leqslant C(n)\,v^{n/(n-2)}(t;l).$$

An integration of the above inequality yields that

$$\begin{aligned}
v^{2/n}(t;l)&\leqslant C(n)\,t+v^{2/n}(0;l)\\
&\leqslant C(n)\,t+\|u_0\|_{n/2},
\end{aligned}$$

which in turn implies that $u(\cdot,t)\in L^{n/2}(R^n)$ for $t>0$ (by letting $l\to\infty$), and

$$\|u(t)\|_{n/2}\leqslant C(n)\,t+\|u_0\|_{n/2}\,.$$

Therefore, by denoting

$$v(t)=\int_{R^n} u^{n/2}(x,t)\,dx,$$

we get

$$v'(t)=-\frac{n(n-2)}{4}\int_{R^n}|\nabla u|^2\,u^{n/2-3}\,dx$$

by taking limit $l \to \infty$ in (21) and (22). On the other hand, by Sobolev imbedding theorem,

$$\int_{R^n} |\nabla u|^2 u^{n/2-3} \, dx = \left(\frac{4}{n-2}\right)^2 \int_{R^n} |\nabla u^{(n-2)/4}|^2 \, dx$$
$$\geqslant c(n) \left(\int_{R^n} u^{n/2} \, dx\right)^{(n-2)/n}$$

Consequently,

$$v'(t) \leqslant -c(n) \, v^{(n-2)/n}(t),$$

where $c(n) > 0$ is a constant. Thus, there exists $T > 0$ such that $v(T) = 0$. We reach a contradiction! This completes the proof of lemma.

Lemma 6. *Let $u_0 > 0$ be as in Lemma 5. Then $u(\cdot, t) \equiv 0$ for $t > 0$.*

Proof. Suppose the contrary. Then Lemma 5 implies that there exists $T > 0$ such that u is positive for $(x, t) \in R^n \times [0, T)$ and $u(\cdot, T) \equiv 0$. Without lose of generality, we assume $\|u_0\|_{n/2} \leqslant \eta$, where η is a small positive number to be fixed later. This is possible since

$$u_\lambda(x, t) = \lambda u\left(x, \frac{t}{\lambda}\right)$$

is a solution of (1) for any $\lambda > 0$. We do the self-similar transformation

$$u(x, t) = (T-t)^\alpha \, v(y, s) \qquad y = \alpha^{1/2} x (T-t)^\beta,$$
$$s = -\alpha \log(T-t) + \alpha \log T,$$

where $\alpha = 1 + 2\beta$, $\beta > 0$ is arbitrary. The equation for v is:

$$v_s = \Delta \log v + k y \cdot \nabla v,$$

where $0 < k = \alpha/\beta < 1/2$. Following the approach of Lemma 5 we derive that

$$w(s) = \int_{R^n} v^{n/2}(x, s) \, dx$$

satisfies

$$w'(s) \leqslant (1-2k) \, w(s) - c(n) \, w(s)^{(n-2)/n},$$

where $c(n) > 0$ is a constant. Now, we choose η sufficiently small so that $w^{2/n}(0) < c(n)/2$. Then we have

$$w'(s) \leqslant -\frac{c(n)}{2}\, w(s)^{(n-2)/n}, \qquad \text{for} \quad s > 0.$$

Again, we conclude that $w(s)$, therefore $v(\cdot, s)$ becomes extinct in finite time. A contradiction. Hence, $u(\cdot, t) \equiv 0$ for $t > 0$.

Proof of Theorem 1. (a) and (b) follow directly from Lemma 2 in Section 2 and Theorem 2 in Section 3. We prove (c) in what follows.

Without lose of generality we assume $\|u_0\|_\infty < \infty$. By maximum principle $\|u(t)\|_\infty < \infty$ for $t > 0$. We shall prove the equivalent assertion that if $u_0 \in L^p(R^n)$ for some $1 < p \leqslant n/2$, then $u(\cdot, t) \equiv 0$ for $t > 0$. Let u_0^* be the radial symmetric re-arrangement of u_0. Then $\|u_0^*\|_p \geqslant \|u_0\|_p$ and $\|u^*(t)\|_p \geqslant \|u(t)\|_p$ by a result of Vazquez, [18]. It is clear that $u^*(x, t)$ is radially symmetric and $u_0^*(x)$ belongs to $L^{n/2}(R^n)$. Hence, by Lemmas 5 and 6, $u^*(\cdot, t) \equiv 0$ for $t > 0$. Consequently, $u(\cdot, t) \equiv 0$ $t > 0$. This completes the proof of Theorem 1.

Remark. By following the exact approach of Lemmas 5 and 6, one can easily prove that if $u_0 \in L^p(R^n)$ for some $p > n/2$, then $u(t) \in L^p(R^n)$ for $t > 0$. Moreover, $\|u(t)\|_p$ is a decreasing function of t. Hence, u is a global solution or becomes extinct in finite time. If we assume that $u_0 \in L^p(R^n)$ for any $p > n/2$, then it can be proved by using interpolation inequality that for any $q > n/2$, $M > 0$,

$$\|u(t)\|_q \leqslant C(1+t)^{-M}.$$

Here C is a constant depends on initial value as well as q and M. But, the open question still remains whether we can demonstrate that the asymptotic spatial profile of u when $t \to \infty$ (in case it is a global solution) or $t \to T$, the extinction time, is characterized by the corresponding self-similar solution given in Section 2 and Section 3. The complexity involved can be illustrated by the family of finite time extinction self-similar solutions which exhibit different rate of extinction as $t \to T$, the extinction time.

Remark. In a forthcoming appear we shall study the more general equation $u_t = \Delta u^m/m$ where $-1 < m < 1$, $m \neq 0$. A corresponding result similar to Theorem 1 will be proved.

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